GRADIENT ESTIMATES FOR A DEGENERATE PARABOLIC EQUATION WITH GRADIENT ABSORPTION AND APPLICATIONS

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Abstract

Qualitative properties of non-negative solutions to a quasilinear degenerate parabolic equation with an absorption term depending solely on the gradient are shown, providing information on the competition between the nonlinear diffusion and the nonlinear absorption. In particular, the limit as $t \to \infty$ of the L^1-norm of integrable solutions is identified, together with the rate of expansion of the support for compactly supported initial data. The persistence of dead cores is also shown. The proof of these results strongly relies on gradient estimates which are first established.

1 Introduction

We investigate the properties of non-negative and bounded continuous solutions to the Cauchy problem

$$\begin{aligned} \partial_t u - \Delta_p u + |\nabla u|^q &= 0\ , \quad (t,x) \in Q_\infty := (0,\infty) \times \mathbb{R}^N\,, && (1.1) \\ u(0) &= u_0 \geq 0\,, \quad x \in \mathbb{R}^N\,, && (1.2) \end{aligned}$$

the parameters p and q ranging in $(2,\infty)$ and $(1,\infty)$, respectively, and the p-Laplacian operator Δ_p being defined by

$$\Delta_p u := \text{ div } \left(|\nabla u|^{p-2}\, \nabla u\right)\,.$$

When $p > 2$, (1.1) is a quasilinear degenerate parabolic equation with a nonlinear absorption term $|\nabla u|^q$ depending solely on the gradient of u, and reduces to the semilinear diffusive Hamilton-Jacobi equation

$$\partial_t v - \Delta v + |\nabla v|^q = 0 \quad \text{in} \quad Q_\infty\,, \tag{1.3}$$

when $p = 2$. Several recent papers have been devoted to the study of properties of non-negative solutions to (1.3) with a particular emphasis on the large time behaviour which turns out to depend strongly on the value of the parameter $q \in (0,\infty)$ [1, 4, 5, 6, 7, 8, 19].

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One of the keystones of these investigations are optimal gradient estimates of the form $\|\nabla(v^\alpha)(t)\|_\infty \le C(\|v(0)\|_\infty)\, t^{-\beta}$ for suitable exponents $\alpha \in (0,1)$ and $\beta > 0$, both depending on N and q [5, 20]. Not only do such estimates provide an instantaneous smoothing effect from $L^\infty(\mathbb{R}^N)$ to $W^{1,\infty}(\mathbb{R}^N)$ but temporal decay estimates as well, the latter being the starting point of a precise study of the large time dynamics. Let us recall here that the proof of the above-mentioned gradient estimates relies on a modification of the Berstein technique [5, 20].

Owing to the nonlinearity of the diffusion term when $p > 2$, the availability of similar gradient estimates for solutions to (1.1), (1.2) is unclear and is actually our first result. More precisely, for $p > 2$ and $q > 1$, we introduce the exponents $\alpha_p \in (0,1)$ and $\beta_{p,q} \in (0,1)$ defined by

$$\frac{1}{\alpha_p} := \frac{p-1}{p-2} - \frac{N-1}{p(N+3)-2(N+1)} \quad \text{and} \quad \beta_{p,q} := \max\left\{\alpha_p, \frac{q-1}{q}\right\}. \tag{1.4}$$

Theorem 1.1 *Consider a non-negative initial condition $u_0 \in \mathcal{BC}(\mathbb{R}^N)$. There is a non-negative viscosity solution $u \in \mathcal{BC}([0,\infty) \times \mathbb{R}^N)$ to (1.1), (1.2) such that*

$$0 \le u(t,x) \le \|u_0\|_\infty\,, \quad (t,x) \in Q_\infty\,, \tag{1.5}$$

$$|\nabla(u^{\alpha_p})(t,x)| \;\le\; C(p,N)\ \|u(s)\|_\infty^{(p\alpha_p+2-p)/p}\ (t-s)^{-1/p}\,, \tag{1.6}$$

$$\left|\nabla\left(u^{\beta_{p,q}}\right)(t,x)\right| \;\le\; C(p,q,N)\ \|u(s)\|_\infty^{(q\beta_{p,q}+1-q)/q}\ (t-s)^{-1/q}\,, \tag{1.7}$$

and

$$\int_{\mathbb{R}^N} (u(t,x)-u(s,x))\ \vartheta(x)\ dx + \int_s^t \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\cdot\nabla\vartheta + |\nabla u|^q\ \vartheta\right)\ dxd\tau = 0 \tag{1.8}$$

for $t > s \ge 0$ and $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

Furthermore, this solution is unique if $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$.

Let us emphasize that the main contribution of Theorem 1.1 is the estimates (1.6), (1.7), and not the existence of a viscosity solution to (1.1) which could probably be obtained by alternative approaches. But, owing to the poor regularity of the solutions to (1.1), (1.2), we cannot prove (1.6) and (1.7) directly and instead use an approximation procedure. Indeed, the proof of (1.6) and (1.7) relies on a modification of the Bernstein technique. It requires the study of the partial differential equation solved by $|\nabla\varphi(u)|^2$ for a suitably chosen function φ and thus some regularity which is not available for solutions to (1.1), (1.2). The existence part of Theorem 1.1 is in fact an intermediate step in the proof of (1.6) and (1.7).

It is clear from (1.6) and (1.7) with $s = 0$ that they lead to different temporal decay estimates. In fact, as we shall see below, (1.6) results from the diffusive part of (1.1) while

(1.7) stems from the absorption term. In particular, it is worth mentioning that (1.6) is also valid for non-negative solutions to the p-Laplacian equation

$$\partial_t w - \Delta_p w = 0 \quad \text{in} \quad Q_\infty \,, \tag{1.9}$$

which seems to be new for $N \ge 2$. When $N = 1$, it has been proved in [17, Theorem 2]. Also, (1.7) is true for non-negative viscosity solutions to the Hamilton-Jacobi equation

$$\partial_t h + |\nabla h|^q = 0 \quad \text{in} \quad Q_\infty \,, \tag{1.10}$$

and can be deduced from [26, Theorem I.1]. For $p = 2$, similar gradient estimates have been obtained in [5, 20] with $\alpha_2 = \beta_{2,q} = (q-1)/q$.

The previous gradient estimates may be improved for non-negative, radially symmetric, and non-increasing initial data.

Theorem 1.2 *Assume that the initial condition $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ is non-negative, radially symmetric, and non-increasing. There is a non-negative viscosity solution u to (1.1), (1.2) satisfying (1.5), (1.8) and such that*

$$x \longmapsto u(t,x) \quad \textit{is non-negative, radially symmetric, and non-increasing,}$$

$$\left|\nabla \left(u^{(p-2)/(p-1)}\right)(t,x)\right| \le C(p,N)\ \|u(s)\|_\infty^{(p-2)/p(p-1)}\ (t-s)^{-1/p} \,, \tag{1.11}$$

$$\left|\nabla \left(u^{(q-1)/q}\right)(t,x)\right| \le \frac{(q-1)^{(q-1)/q}}{q}\ t^{-1/q} \quad \textit{if} \quad q \ge p-1 \,, \tag{1.12}$$

and

$$\left|\nabla \left(u^{(p-2)/(p-1)}\right)(t,x)\right| \le C(p,q)\ \|u(s)\|_\infty^{(p-1-q)/q(p-1)}\ (t-s)^{-1/q} \quad \textit{if} \quad q \in (1,p-1) \,, \tag{1.13}$$

for $t > s \ge 0$.

Theorem 1.2 is proved as Theorem 1.1 for $N = 1$. We will thus only give the proof of the latter.

Here again, the gradient estimate (1.11) is valid for non-negative solutions to the p-Laplacian equation (1.9) with radially symmetric and non-increasing initial data and is easily seen to be optimal in that case: indeed, the Barenblatt solution to the p-Laplacian equation (1.9) is given by

$$\mathcal{B}(t,x) = t^{-N\eta} \left(1 - \gamma_p \left(\frac{|x|}{t^\eta}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)} \,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \,,$$

(see, e.g., [16, Ch. XI, Eq. (1.6)]) and $\nabla\left(\mathcal{B}^\vartheta\right)(t,x)$ is bounded only for $\vartheta \ge (p-2)/(p-1)$.

Remark 1.3 *Since we are mainly interested in qualitative properties of solutions to (1.1), (1.2), we leave aside the question of uniqueness of such solutions for initial data in* $\mathcal{BC}(\mathbb{R}^N)\setminus\mathcal{BUC}(\mathbb{R}^N)$. *Nevertheless, since the solutions in Theorems 1.1 and 1.2 are constructed as limits of classical solutions, they still enjoy a comparison principle. More precisely, if* u_0 *and* $\hat{u}_0$ *are two non-negative functions in* $\mathcal{BC}(\mathbb{R}^N)$ *such that* $u_0 \le \hat{u}_0$, *then the corresponding solutions* u *and* $\hat{u}$ *to (1.1) with initial data* u_0 *and* $\hat{u}_0$ *constructed in Theorem 1.1 satisfy* $u(t,x) \le \hat{u}(t,x)$ *for all* $(t,x) \in Q_\infty$. *This fact will be used repeatedly in the sequel.*

Several qualitative properties follow from the previous gradient estimates. As a first consequence, we derive temporal decay estimates in $W^{1,\infty}(\mathbb{R}^N)$ for non-negative and integrable solutions to (1.1), (1.2). We set

$$q_* := p - \frac{N}{N+1}\,, \quad \xi := \frac{1}{q(N+1)-N}\,, \quad \eta := \frac{1}{N(p-2)+p}\,. \tag{1.14}$$

Proposition 1.4 *Assume that*

$$u_0 \in L^1(\mathbb{R}^N) \cap \mathcal{BC}(\mathbb{R}^N)\,, \quad u_0 \ge 0\,, \tag{1.15}$$

and denote by u *the corresponding viscosity solution to (1.1), (1.2) constructed in Theorem 1.1. Then* $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N))$.

Let $t > 0$. *If* $q \in (1, q_*)$, *then*

$$\|u(t)\|_\infty \le C\, \|u_0\|_1^{q\xi}\, t^{-N\xi}\,, \tag{1.16}$$

$$\|\nabla u(t)\|_\infty \le C\, \|u_0\|_1^{\xi}\, t^{-(N+1)\xi}\,, \tag{1.17}$$

while, if $q > q_*$,

$$\|u(t)\|_\infty \le C\, \|u_0\|_1^{p\eta}\, t^{-N\eta}\,, \tag{1.18}$$

$$\|\nabla u(t)\|_\infty \le C\, \|u_0\|_1^{2\eta}\, t^{-(N+1)\eta}\,. \tag{1.19}$$

Recall that the L^∞-norm of non-negative and integrable solutions w to the p-Laplacian equation (1.9) decays as $t^{-N\eta}$ [22, Theorem 3]. However this decay might be enhanced by the nonlinear absorption term and this is indeed the case for $q \in (1, q_*)$. Indeed, $t^{-N\xi} \le t^{-N\eta}$ for $t \ge 1$ and $q \in (1, q_*)$. According to Proposition 1.4, we thus expect the nonlinear absorption term to be negligible as $t \to \infty$ for $q > q_*$ and the large time dynamics to feel the effects of the absorption only for $q \in (1, q_*)$. The next result is a further step in that direction.

It readily follows from (1.1) and the non-negativity of u that $t \longmapsto \|u(t)\|_1$ is a non-increasing and non-negative function. Introducing

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 = \inf_{t\ge 0} \left\{\|u(t)\|_1\right\} \in [0, \|u_0\|_1]\,, \tag{1.20}$$

we study the possible values of $I_1(\infty)$.

Proposition 1.5 *Assume that u_0 satisfies (1.15) with $\|u_0\|_1 > 0$ and denote by u the corresponding viscosity solution to (1.1), (1.2) constructed in Theorem 1.1. Then $I_1(\infty) > 0$ if and only if $q > q_*$, the parameter q_* being defined in (1.14).*

Since $\|w(t)\|_1 = \|w(0)\|_1$ for all $t \geq 0$ for non-negative and integrable solutions w to the p-Laplacian equation (1.9), we realize that the absorption term is not strong enough for $q > q_*$ to drive the L^1-norm of $u(t)$ to zero as $t \to \infty$, thus indicating a diffusion-dominated behaviour for large times. For $q \in (p-1, p)$ Proposition 1.5 is already proved in [1, Theorems 1.3 & 1.4] by a different method.

We next turn to a property which marks a striking difference between the semilinear case $p = 2$ and the quasilinear case $p > 2$ corresponding to *slow diffusion*, namely the finite speed of propagation. Since the support of non-negative and compactly supported solutions w to the p-Laplacian equation (1.9) grows as t^η, it is natural to wonder whether the absorption term will slow down this process.

Theorem 1.6 *Assume that u_0 fulfils (1.15) and is compactly supported, and denote by u the corresponding solution to (1.1), (1.2). For $t \geq 0$ we put*

$$\varrho(t) := \inf \{R > 0 \quad \text{such that} \quad u(t,x) = 0 \quad \text{for} \quad |x| > R\} . \tag{1.21}$$

Then $\varrho(t) < \infty$ for all $t \geq 0$ and:

(i) If $q \in (1, p-1)$ then

$$\limsup_{t \to \infty} \varrho(t) < \infty . \tag{1.22}$$

(ii) If $q = p-1$ then

$$\varrho(t) \leq C \ (1 + \ln t) \quad \text{for} \quad t \geq 1 . \tag{1.23}$$

(iii) If $q \in (p-1, q_*)$ then

$$\varrho(t) \leq C \ t^{(q-p+1)/(2q-p)} \quad \text{for} \quad t \geq 1 . \tag{1.24}$$

(iv) If $q \geq q_*$ then

$$\varrho(t) \leq C \ t^{\eta} \quad \text{for} \quad t \geq 1 . \tag{1.25}$$

Here again, the absorption term seems to have no real effect on the expansion on the support of $u(t)$ for $q > q_*$ as the upper bound (1.25) is exactly the growth rate of the support for non-negative and compactly supported solutions w to the p-Laplacian equation (1.9). But, as soon as q is below q_*, the dynamics starts to feel the effects of the absorption term and the expansion of the support of $u(t)$ slows down. It even stops for $q \in (1, p-1)$. In that case, the support of $u(t)$ remains *localized* in a fixed ball of $\mathbb{R}^N$: such a property is already enjoyed by compactly supported non-negative solutions to second-order degenerate parabolic equations with a sufficiently strong absorption involving the solution only as, for

instance, $\partial_t z - \Delta_p z + z^r = 0$ in Q_∞ when $r \in (1, p-1)$ [15, 23, 28]. It has apparently remained unnoticed for second-order degenerate parabolic equations with an absorption term depending solely on the gradient. In our case, this property is clearly reminiscent of that enjoyed by the solutions h to the Hamilton-Jacobi equation (1.10): namely, the support of $h(t)$ does not evolve through time evolution [2]. Finally, for $q \in (p-1, q_*)$, compactly supported self-similar solutions to (1.1) are constructed and the boundaries of their support evolve at the speed given by the right-hand side of (1.24).

As a by-product of the proof of Theorem 1.6 we obtain improved decay estimates for the L^1-norm of solutions to (1.1), (1.2) with compactly supported initial data.

Corollary 1.7 *Assume that u_0 fulfils (1.15) and is compactly supported. Then*

(i) If $q \in (1, p-1)$ then

$$\|u(t)\|_1 \le C\ t^{-1/(q-1)}\,, \quad t \ge 2\,. \tag{1.26}$$

(ii) If $q = p-1$ then

$$\|u(t)\|_1 \le C\ t^{-1/(q-1)}\ (\ln t)^{1/\xi(q-1)} \quad \textit{for} \quad t \ge 2\,. \tag{1.27}$$

(iii) If $q \in (p-1, q_*)$ then

$$\|u(t)\|_1 \le C\ t^{-((N+1)(q_*-q))/(2q-p)} \quad \textit{for} \quad t \ge 2\,. \tag{1.28}$$

(iv) If $q = q_*$ then

$$\|u(t)\|_1 \le C\ (\ln t)^{-1/(q-1)} \quad \textit{for} \quad t \ge 2\,. \tag{1.29}$$

For $q \in (p-1, q_*]$, Theorem 1.6 and Corollary 1.7 are already proved in [1, Theorems 1.1 & 1.2] by a completely different approach. In addition, for non-compactly supported initial data, temporal decay estimates involving the behaviour of u_0 for large values of x are obtained in [1, Theorem 1.3] for the L^1-norm of u. Let us also mention that the decay rate of $\|u(t)\|_1$ for $q \in (1, p-1)$ is the same as the one obtained in [2] for non-negative and compactly supported solutions to the Hamilton-Jacobi equation (1.10). The bound (1.26) then provides another clue of the dominance of the absorption term for $q \in (1, p-1)$. That it is indeed true is shown in [25].

For $q \in (1, p-1)$, it follows from Theorem 1.6 (i) that the support of the solutions to (1.1), (1.2) with compactly supported initial data remains bounded through time evolution. A natural counterpart of this phenomenon is to study what happens to a solution to (1.1), (1.2) starting from an initial condition vanishing inside a ball of $\mathbb{R}^N$. It turns out that, if the radius of the ball is sufficiently large, the solution still vanishes inside of a smaller ball for all times, a phenomenon which may be called the persistence of *dead cores*.

Proposition 1.8 *Consider a non-negative initial condition $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ such that*

$$u_0(x) = 0 \quad \text{if} \quad |x| \le R_0 \tag{1.30}$$

for some $R_0 > 0$, and denote by u the corresponding solution to (1.1), (1.2) constructed in Theorem 1.1. If $q \in (1, p-1)$ there is a constant $\delta_0 = \delta_0(p,q) > 0$ such that, if $R_0 \ge \delta_0 \ \|u_0\|_\infty^{(p-1-q)/(p-q)}$ then

$$u(t,x) = 0 \quad \text{if} \quad |x| \le R_0 - \delta_0 \ \|u_0\|_\infty^{(p-1-q)/(p-q)} \quad \text{and} \quad t \ge 0\,.$$

The proof of Proposition 1.8 is in fact quite similar to that of Theorem 1.6 (i).

This paper is organized as follows: gradient estimates for an approximation of (1.1) are established in Section 2 by a modified Bernstein technique with the help of a trick introduced in [10] to obtain gradient estimates for the porous medium equation. Theorems 1.1 and 1.2 are then proved in Section 3. Sections 4 and 5 are devoted to integrable initial data for which we prove Propositions 1.4 and 1.5. We focus on compactly supported initial data in Section 6 where Theorem 1.6 and Corollary 1.7 are proved. The persistence of dead cores is studied in Section 7 while the proof of a technical lemma from Section 2 is postponed to the appendix.

2 Gradient estimates

As already mentioned the proof of the gradient estimates (1.6) and (1.7) rely on a modified Bernstein technique: owing to the degeneracy of the diffusion we cannot expect (1.1) to have smooth solutions and we thus need to use an approximation procedure. We first report the following technical lemma.

Lemma 2.1 *Let a and b be two non-negative functions in $\mathcal{C}^2([0,\infty))$ and u be a classical solution to*

$$\partial_t u - \ \text{div} \ \left(a\left(|\nabla u|^2\right) \ \nabla u\right) + b\left(|\nabla u|^2\right) = 0 \quad \text{in} \quad Q_\infty\,. \tag{2.1}$$

Consider next a $\mathcal{C}^3$-smooth increasing function φ and set $v := \varphi^{-1}(u)$ and $w := |\nabla v|^2$. Then w satisfies the following differential inequality

$$\partial_t w - \mathcal{A}w - \mathcal{V}\cdot\nabla w + 2\ \mathcal{R}_1\ w^2 + 2\ \mathcal{R}_2\ w \le 0 \quad \text{in} \quad Q_\infty\,, \tag{2.2}$$

where $\mathcal{A}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by

$$\mathcal{A}w := a\ \Delta w + 2a'\ (\nabla u)^t D^2 w \nabla u\,, \tag{2.3}$$

$$\mathcal{R}_1 := -a\ \left(\frac{\varphi''}{\varphi'}\right)' - \left((N-1)\ \frac{a'^2}{a} + 4\ a''\right)\ (\varphi'\varphi'')^2\ w^2 - 2\ a'\ w\left(2\varphi''^2 + \varphi'\varphi'''\right)\,, \tag{2.4}$$

$$\mathcal{R}_2 := \frac{\varphi''}{\varphi'^2}\ \left(2\ b'\ \varphi'^2\ w - b\right)\,, \tag{2.5}$$

while $\mathcal{V}$ is given by (A.2) below. Here and in the following we omit the variable in a, b and φ and their derivatives.

Furthermore, if φ is convex, a is non-decreasing and $x \longmapsto u(t,x)$ is radially symmetric and non-increasing for each $t \geq 0$, then $\mathcal{R}_1$ may be replaced by $\mathcal{R}_1^r$ given by

$$\mathcal{R}_1^r := -a \ \left(\frac{\varphi''}{\varphi'}\right)' - 4 \ a'' \ (\varphi'\varphi'')^2 \ w^2 - 2 \ a' \ w \left(2\varphi''^2 + \varphi'\varphi'''\right) , \tag{2.6}$$

The proof of Lemma 2.1 is rather technical and is postponed to the appendix. We however emphasize that it uses a trick introduced by Bénilan [10] to prove gradient estimates for solutions to the porous medium equation in several space dimensions. It is also worth noticing that $\mathcal{R}_1 = \mathcal{R}_1^r$ for $N = 1$.

Consider next a non-negative function $u_0 \in \mathcal{BC}(\mathbb{R}^N)$. There is a sequence of functions $(u_{0,k})_{k\geq 1}$ such that, for each integer $k \geq 1$, $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$,

$$0 \leq u_{0,k}(x) \leq u_{0,k+1}(x) \leq u_0(x) , \quad x \in \mathbb{R}^N , \tag{2.7}$$

and $(u_{0,k})$ converges uniformly towards u_0 on compact subsets of $\mathbb{R}^N$. In addition, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ we may assume that

$$\|\nabla u_{0,k}\|_\infty \leq \left(1 + \frac{K_1}{k}\right) \ \|\nabla u_0\|_\infty , \tag{2.8}$$

for some constant $K_1 > 0$ depending only on the approximation process. Next, since $\xi \longmapsto |\xi|^{p-2}$ and $\xi \longmapsto |\xi|^q$ are not regular enough for small values of p and q, we set

$$a_\varepsilon(\xi) := \left(\varepsilon^2 + \xi\right)^{(p-2)/2} \quad \text{and} \quad b_\varepsilon(\xi) := \left(\varepsilon^2 + \xi\right)^{q/2} - \varepsilon^q , \quad \xi \geq 0 , \tag{2.9}$$

for $\varepsilon \in (0, 1/2)$. Then, given

$$0 < \gamma \leq \min\left\{\frac{3}{4}, 2\beta_{p,q}, q, \frac{q+2}{2}\right\} , \tag{2.10}$$

the Cauchy problem

$$\begin{aligned} \partial_t u_{k,\varepsilon} - \ \text{div} \ \left(a_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) \ \nabla u_{k,\varepsilon}\right) + b_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) \ &= \ 0 , \quad (t,x) \in Q_\infty , &(2.11)\\ u_{k,\varepsilon}(0) \ &= \ u_{0,k} + \varepsilon^\gamma , \quad x \in \mathbb{R}^N , &(2.12)\end{aligned}$$

has a unique classical solution $u_{k,\varepsilon} \in \mathcal{C}^{(3+\delta)/2, 3+\delta}([0,\infty) \times \mathbb{R}^N)$ for some $\delta \in (0,1)$ [24]. Observing that ε^γ and $\|u_0\|_\infty + \varepsilon^\gamma$ are solutions to (2.11) with $\varepsilon^\gamma \leq u_{k,\varepsilon}(0,x) \leq \|u_0\|_\infty + \varepsilon^\gamma$, the comparison principle warrants that

$$\varepsilon^\gamma \leq u_{k,\varepsilon}(t,x) \leq \|u_0\|_\infty + \varepsilon^\gamma , \quad (t,x) \in [0,\infty) \times \mathbb{R}^N . \tag{2.13}$$

We now turn to estimates on the gradient of $u_{k,\varepsilon}$ and first point out that, thanks to the regularity of a_ε, b_ε and $u_{k,\varepsilon}$, we may use Lemma 2.1. We first take $\varphi(r)=\varphi_0(r):=r$ for $r\ge 0$ so that $w=|\nabla u_{k,\varepsilon}|^2$ and $\mathcal{R}_1=\mathcal{R}_2=0$. Therefore w satisfies

$$\partial_t w-\mathcal{A}w-\mathcal{V}\cdot\nabla w\le 0 \quad \text{in} \quad Q_\infty\,.$$

Since $w(0)\le \|\nabla u_{0,k}\|_\infty^2$ the comparison principle ensures that

$$\|\nabla u_{k,\varepsilon}(t)\|_\infty\le \|\nabla u_{0,k}\|_\infty\,, \quad t\ge 0\,. \tag{2.14}$$

We now establish gradient estimates similar to (1.6) and (1.7) for $u_{k,\varepsilon}$. We first use the specific choice of a_ε and b_ε to compute $\mathcal{R}_1$ and $\mathcal{R}_2$.

Lemma 2.2 *Introducing $g:=\left(|\nabla u_{k,\varepsilon}|^2+\varepsilon^2\right)^{1/2}$, we have*

$$\mathcal{R}_1=-(p-1)\ g^{p-2}\ \left\{\left(\frac{\varphi''}{\varphi'}\right)'+\frac{\alpha_p}{1-\alpha_p}\ \left(\frac{\varphi''}{\varphi'}\right)^2\right\}+\varepsilon^2\ \mathcal{R}_{11} \tag{2.15}$$

with

$$\begin{aligned}\mathcal{R}_{11} &= (p-2)\ \left(\frac{\varphi''}{\varphi'}\right)'\ g^{p-4}+\frac{(p-2)(p(N+3)-2(N+1))}{4}\ \left(\frac{\varphi''}{\varphi'}\right)^2\ g^{p-4}\\ &+\ \frac{(p-2)(p(N+3)-2(N+7))}{4}\ \left(\frac{\varphi''}{\varphi'}\right)^2\ \left(g^2-\varepsilon^2\right)\ g^{p-6}\,,\end{aligned}$$

and

$$\mathcal{R}_2=\frac{\varphi''}{\varphi'^2}\ \left\{(q-1)\ g^q+\varepsilon^q-q\ \varepsilon^2\ g^{q-2}\right\}\,. \tag{2.16}$$

After these preliminary computations we are in a position to state and prove the main result of this section.

Proposition 2.3 *There are positive real numbers $C=C(p,N)$ and $D_1(k)=D_1(k,p,N)$ such that, for $\varepsilon\in(0,1/2)$, $x\in\mathbb{R}^N$, and $t\in\left(0,\varepsilon^{-1/4}\right)$,*

$$\left|\nabla\left(u_{k,\varepsilon}^{\alpha_p}\right)(t,x)\right|\le C\ \left(1+D_1(k)\ \varepsilon^{1/4}\right)^{2/p}\ \left(\|u_{0,k}\|_\infty+\varepsilon^\gamma\right)^{(p\alpha_p+2-p)/p}\ t^{-1/p}\,. \tag{2.17}$$

There are a positive real number $D_2(k)=D_2(k,p,q,N)$ and a positive function $\omega\in\mathcal{C}([0,\infty))$ such that $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$ and

$$\begin{aligned}\left|\nabla\left(u_{k,\varepsilon}^{\beta_{p,q}}\right)(t,x)\right| \ &\le\ \frac{\beta_{p,q}}{(q-1)^{1/q}(1-\beta_{p,q})^{1/q}}\ \left(\frac{1}{q}+D_2(k)\ \omega(\varepsilon)^{1/2}\right)^{1/q}\\ &\quad\times\left(\|u_{0,k}\|_\infty+\varepsilon^\gamma\right)^{(q\beta_{p,q}+1-q)/q}\ \ t^{-1/q}\end{aligned} \tag{2.18}$$

for $t\in(\left(0,\omega(\varepsilon)^{-1/2}\right)$, $x\in\mathbb{R}^N$, and $\varepsilon\in(0,\min\{q-1,1/2\})$.

The proof of Proposition 2.3 relies on suitable choices of the function φ in $\mathcal{R}_1$ and $\mathcal{R}_2$. To motivate the forthcoming choices, we first note that, if $\varphi(r) = r^{1/\alpha_p}$, then $\mathcal{R}_1 = \varepsilon^2\ \mathcal{R}_{11}$ and (2.17) will in fact be obtained by choosing a "small perturbation" of $r \mapsto r^{1/\alpha_p}$, namely $\varphi(r) = \varphi_1(r) := \left(2Kr - r^2\right)^{1/\alpha_p}$ for K sufficiently large. Such a choice has already been employed for the p-Laplacian equation in one space dimension $N = 1$ for the same purpose [17]. Next, previous investigations for the case $p = 2$ suggest that $\varphi(r) = r^{q/(q-1)}$ is a suitable choice in $\mathcal{R}_2$ [5]. However, with this choice of φ, $\mathcal{R}_1$ might give a non-positive contribution according to the value of p and a suitable choice turns out to be $\varphi(r) = \varphi_2(r) := \beta_{p,q}\ r^{1/\beta_{p,q}}$.

Proof of Proposition 2.3. We first establish (2.17). Consider $\mu > 0$ to be specified later and put

$$K := \sqrt{1+\mu}\ M^{\alpha_p}\,, \quad M := \|u_{0,k}\|_\infty + \varepsilon^\gamma$$

and $\varphi_1(r) := \left(2Kr - r^2\right)^{1/\alpha_p}$ for $r \in [0, K]$. Then v is given by

$$v := K - \left(K^2 - u_{k,\varepsilon}^{\alpha_p}\right)^{1/2} \tag{2.19}$$

and satisfies

$$\frac{\varepsilon^{\gamma\alpha_p}}{2K} \le v \le K - \left(K^2 - M^{\alpha_p}\right)^{1/2} \le M^{\alpha_p/2} \tag{2.20}$$

by (2.13). Thanks to the bounds (2.20), we can find μ large enough such that φ_1 enjoys the following properties:

$$0 \ge \left(\frac{\varphi_1''}{\varphi_1'}\right)'(v) \ge -\frac{C_1(\mu)}{v^2}\,, \tag{2.21}$$

$$0 \le \frac{\varphi_1''}{\varphi_1'}(v) \le \frac{C_2(\mu)}{v}\,, \tag{2.22}$$

$$\left(\frac{\varphi_1''}{\varphi_1'}\right)'(v) + \frac{\alpha_p}{1-\alpha_p}\ \left(\frac{\varphi_1''}{\varphi_1'}\right)^2(v) \le -\frac{1+\alpha_p}{2\alpha_p}\ \frac{1}{Kv}\,. \tag{2.23}$$

We then infer from (2.21) and (2.22) that

$$\mathcal{R}_{11} \ge -\frac{C_3(\mu)}{v^2}\ g^{p-4}\,.$$

Therefore, by (2.20) and the elementary inequality $g \ge |\nabla u_{k,\varepsilon}|$, we have

$$w\ \mathcal{R}_{11} \ge -\frac{|\nabla u_{k,\varepsilon}|^2}{(\varphi_1')^2(v)}\ \frac{C_3(\mu)}{v^2}\ g^{p-4} \ge -\frac{C_4(\mu)}{M\ v^{2/\alpha_p}}\ g^{p-2} \ge -\frac{C_5(\mu)}{\varepsilon^{2\gamma}}\ g^{p-2}\,.$$

Combining the previous inequality with (2.15) and (2.23), we obtain

$$w^2\ \mathcal{R}_1 \ge \frac{C_6(\mu)\ M^{-\alpha_p/2}}{v}\ g^{p-2}\ w^2 - C_5(\mu)\ \varepsilon^{2(1-\gamma)}\ g^{p-2}\ w\,.$$

Now, we have $g \leq \|\nabla u_{0,k}\|_\infty + \varepsilon$ by (2.14) and

$$g^2 \geq |\nabla u_{k,\varepsilon}|^2 = (\varphi_1')^2(v) \geq C_7(\mu)\ M\ v^{2(1-\alpha_p)/\alpha_p}\ w$$

by (2.20). The previous lower bound for $w^2\ \mathcal{R}_1$ then gives

$$w^2\ \mathcal{R}_1 \geq \frac{C_8(\mu)\ M^{(p-2-\alpha_p)/2}}{v^{((p-1)\alpha_p-(p-2))/\alpha_p}}\ w^{(p+2)/2} - C_9(\mu,k)\ \varepsilon^{2(1-\gamma)}\ w\,.$$

Since $(p-1)\alpha_p \geq (p-2)$ and $v \leq M^{\alpha_p/2}$ by (2.20), we end up with

$$w^2\ \mathcal{R}_1 \geq C_{10}(\mu)\ M^{(2(p-2)-p\alpha_p)/2}\ w^{(p+2)/2} - C_9(\mu,k)\ \varepsilon^{2(1-\gamma)}\ w\,. \tag{2.24}$$

Next, since $q > 1$ and $g \geq \varepsilon$, we infer from the monotonicity of φ_1 and (2.22) that $\mathcal{R}_2 \geq 0$. Recalling (2.2) and (2.24) we have shown that

$$\mathcal{L}_1 w := \partial_t w - \mathcal{A} w - \mathcal{V}\cdot\nabla w + 2\ C_{10}(\mu)\ M^{(2(p-2)-p\alpha_p)/2}\ w^{(p+2)/2} - 2\ C_9(\mu,k)\ \varepsilon^{2(1-\gamma)}\ w \leq 0$$

in Q_∞. It is then straightforward to check that

$$S_1(t) := \left(\frac{1+2\ C_9(\mu,k)\ \varepsilon^{1/4}}{p\ C_{10}(\mu)}\right)^{2/p}\ M^{(p\alpha_p-2(p-2))/p}\ t^{-2/p}$$

satisfies $\mathcal{L}_1 S_1 \geq 0$ in $\left(0,\varepsilon^{-1/4}\right)\times\mathbb{R}^N$. The comparison principle then ensures that $w(t,x) \leq S_1(t)$ for $(t,x) \in \left(0,\varepsilon^{-1/4}\right)\times\mathbb{R}^N$. The estimate (2.17) then readily follows with the help of (2.20).

To prove (2.18) we take $\varphi_2(r) := \beta_{p,q}\ r^{1/\beta_{p,q}}$, so that $v = (u/\beta_{p,q})^{\beta_{p,q}}$ satisfies

$$\frac{\varepsilon^{\gamma\beta_{p,q}}}{\beta_{p,q}^{\beta_{p,q}}} \leq v \leq \frac{M^{\beta_{p,q}}}{\beta_{p,q}^{\beta_{p,q}}} \quad \text{with} \quad M := \|u_{0,k}\|_\infty + \varepsilon^\gamma\,, \tag{2.25}$$

by (2.13). Concerning $\mathcal{R}_1$, the computations are much simpler than in the previous case and it follows from the definition of $\beta_{p,q}$ and (2.14) that

$$\begin{aligned} w^2\ \mathcal{R}_1 &\geq C_{11}\ \frac{\beta_{p,q}-\alpha_p}{\alpha_p\beta_{p,q}}\ \frac{g^{p-2}\ w^2}{v^2} - C_{12}\ \varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}}\ g^{p-2}\ w \\ w^2\ \mathcal{R}_1 &\geq -C_{13}(k)\ \varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}}\ w\,. \end{aligned} \tag{2.26}$$

For $\mathcal{R}_2$, we first claim that

$$(q-1)\ g^q + \varepsilon^q - q\ \varepsilon^2\ g^{q-2} \geq (q-1-\varepsilon)\ g^q - C_{14}\ \left(\varepsilon^{(q+2)/2} + \varepsilon^q\right)\,. \tag{2.27}$$

Indeed, if $q > 2$, it follows from the Young inequality that

$$\begin{aligned} (q-1)\ g^q + \varepsilon^q - q\ \varepsilon^2\ g^{q-2} &\geq (q-1)\ g^q - \varepsilon\ g^q - 2\ (q-2)^{(q-2)/2}\ \varepsilon^{(q+2)/2} \\ &\geq (q-1-\varepsilon)\ g^q - 2\ (q-2)^{(q-2)/2}\ \varepsilon^{(q+2)/2}\,. \end{aligned}$$

If $q \in (1,2]$, we have

$$(q-1)\ g^q + \varepsilon^q - q\ \varepsilon^2\ g^{q-2} \geq (q-1)\ g^q + \varepsilon^q - q\ \varepsilon^q \geq (q-1-\varepsilon)\ g^q - (q-1)\ \varepsilon^q\,,$$

which completes the proof of (2.27). We then infer from (2.16), (2.25), and (2.27) that

$$\begin{aligned}
\mathcal{R}_2 &\geq \frac{1-\beta_{p,q}}{\beta_{p,q}}\ \frac{1}{v^{1/\beta_{p,q}}}\ \left[(q-1-\varepsilon)\ (\varphi_2')^q(v)\ w^{q/2} - C_{14}\ \left(\varepsilon^{(q+2)/2} + \varepsilon^q\right)\right] \\
&\geq \frac{1-\beta_{p,q}}{\beta_{p,q}}\ (q-1-\varepsilon)\ v^{(q(1-\beta_{p,q})-1)/\beta_{p,q}}\ w^{q/2} - C_{15}\ \left(\varepsilon^{(q+2-2\gamma)/2} + \varepsilon^{q-\gamma}\right) \\
&\geq \frac{1-\beta_{p,q}}{\beta_{p,q}^{q(1-\beta_{p,q})}}\ (q-1-\varepsilon)\ M^{q(1-\beta_{p,q})-1}\ w^{q/2} - C_{15}\ \left(\varepsilon^{(q+2-2\gamma)/2} + \varepsilon^{q-\gamma}\right)\,,
\end{aligned}$$

Recalling (2.26) we have thus shown that w satisfies

$$\begin{aligned}
\mathcal{L}_2 w &:= \partial_t w - \mathcal{A}w - \mathcal{V}\cdot\nabla w + 2\ \frac{1-\beta_{p,q}}{\beta_{p,q}^{q(1-\beta_{p,q})}}\ (q-1-\varepsilon)\ M^{q(1-\beta_{p,q})-1}\ w^{(q+2)/2} \\
&- \quad C_{16}(k)\ \omega(\varepsilon)\ w \leq 0
\end{aligned}$$

in Q_∞, where $\omega(\varepsilon) := \varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}} + \varepsilon^{(q+2-2\gamma)/2} + \varepsilon^{q-\gamma} \longrightarrow 0$ as $\varepsilon \to 0$ by the choice (2.10) of γ. The function

$$S_2(t) := \frac{\beta_{p,q}^{2(1-\beta_{p,q})}}{2^{2/q}\ (1-\beta_{p,q})^{2/q}\ (q-1-\varepsilon)^{2/q}}\ \left(\frac{2+q\ C_{16}(k)\ \omega(\varepsilon)^{1/2}}{q}\right)^{2/q}\ M^{2(1-q(1-\beta_{p,q}))/q}\ t^{-2/q}$$

satisfies $\mathcal{L}_2 S_2 \geq 0$ in $\left(0,\omega(\varepsilon)^{-1/2}\right)\times\mathbb{R}^N$. We then deduce from the comparison principle that $w(t,x) \leq S_2(t)$ for $(t,x) \in \left(0,\omega(\varepsilon)^{-1/2}\right)\times\mathbb{R}^N$. The estimate (2.18) then readily follows. $\square$

3 Existence

We are now in a position to prove Theorem 1.1 and proceed along the lines of [20].
Step 1: $\varepsilon \to 0$. We first let $\varepsilon \to 0$. For that purpose, we observe that the gradient bound (2.14) and (2.11) imply the time equicontinuity of $(u_{k,\varepsilon})_{\varepsilon>0}$.

Lemma 3.1 *For $k \geq 1$, $\varepsilon > 0$, $x \in \mathbb{R}^N$, $t_1 \geq 0$, and $t_2 > t_1$, we have*

$$|u_{k,\varepsilon}(t_2,x) - u_{k,\varepsilon}(t_1,x)| \leq C\ \left(\|\nabla u_{0,k}\|_\infty + \|\nabla u_{0,k}\|_\infty^{p-1}\right)\ (t_2-t_1)^{1/2} + \|\nabla u_{0,k}\|_\infty^q\ (t_2-t_1)\,.$$

The proof of Lemma 3.1 is similar to that of [20, Lemma 5] to which we refer.

We next fix $k \geq 1$. Owing to (2.13), (2.14), and Lemma 3.1, we may apply the Arzelà-Ascoli theorem to obtain a subsequence of $(u_{k,\varepsilon})_{\varepsilon>0}$ (not relabeled) and a non-negative function $u_k \in \mathcal{BC}([0,\infty)\times\mathbb{R}^N)$ such that

$$u_{k,\varepsilon} \longrightarrow u_k \quad \text{uniformly on any compact subset of} \quad [0,\infty)\times\mathbb{R}^N\,. \tag{3.1}$$

Furthermore, as $u_{k,\varepsilon}$ is a classical solution to (2.11), (2.12), the classical stability result for continuous viscosity solutions allows us to conclude that u_k is a viscosity solution to (1.1) with initial condition $u_{0,k}$ (see, e.g., [13, Theorem 1.4] or [3, Théorème 2.3]). By (3.1) and weak convergence arguments, we next infer from (2.13), (2.17), and (2.18) that

$$\begin{aligned} 0 \le u_k(t,x) &\le \|u_0\|_\infty\,, &(3.2)\\ \left|\nabla\left(u_k^{\alpha_p}\right)(t,x)\right| &\le C\ \|u_{0,k}\|_\infty^{(p\alpha_p+2-p)/p}\ t^{-1/p}\,, &(3.3)\end{aligned}$$

$$\left|\nabla\left(u_k^{\beta_{p,q}}\right)(t,x)\right| \le \frac{\beta_{p,q}}{(q^2-q)^{1/q}(1-\beta_{p,q})^{1/q}}\ \|u_{0,k}\|_\infty^{(q\beta_{p,q}+1-q)/q}\ t^{-1/q} \tag{3.4}$$

for all $(t,x)\in Q_\infty$. Finally, (2.11) also reads

$$\partial_t u_{k,\varepsilon} - \text{ div }\left(|\nabla u_{k,\varepsilon}|^{p-2}\ \nabla u_{k,\varepsilon}\right) = \text{ div }\left(f_{k,\varepsilon}\right) + g_{k,\varepsilon} \quad\text{in}\quad Q_\infty$$

with

$$f_{k,\varepsilon} := \left\{a_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) - |\nabla u_{k,\varepsilon}|^{p-2}\right\}\ \nabla u_{k,\varepsilon} \quad\text{and}\quad g_{k,\varepsilon} := -b_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right)\,.$$

It follows from the definition of a_ε and (2.14) that $(g_{k,\varepsilon})$ is bounded in $L^\infty(Q_\infty)$ and $(f_{k,\varepsilon})$ converges to zero in $L^\infty(Q_\infty)$ as $\varepsilon\to 0$. We may then apply [12, Theorem 4.1] to conclude that

$$\nabla u_{k,\varepsilon} \longrightarrow \nabla u_k \quad\text{a.e. in}\quad Q_\infty\,. \tag{3.5}$$

Consequently, upon extracting a further subsequence, we may assume that

$$\nabla u_{k,\varepsilon} \longrightarrow \nabla u_k \quad\text{a.e. in}\quad L^r((0,T)\times B(0,R)) \tag{3.6}$$

for every $r\in[1,\infty)$, $T>0$, and $R>0$. It then readily follows that u_k satisfies (1.8) with $u_{0,k}$ instead of u_0.

Step 2: $k\to\infty$. It remains to pass to the limit as $k\to\infty$. To this end we first observe that (2.7) implies that $u_{0,k}(x) - u_{0,k+1}(y) \le \|\nabla u_{0,k}\|_\infty\ |y-x|$ for $k\ge 1$, $x\in\mathbb{R}^N$, and $y\in\mathbb{R}^N$. It then follows from the comparison principle [18, Theorem 2.1] that

$$u_k(t,x) \le u_{k+1}(t,x) \quad\text{for } (t,x)\in Q_\infty \quad\text{and}\quad k\ge 1\,. \tag{3.7}$$

Therefore, by (2.7), (3.2), and (3.7), the function

$$u(t,x) := \sup_{k\ge 1} u_k(t,x) \in [0,\|u_0\|_\infty] \tag{3.8}$$

is well-defined for $(t,x)\in[0,\infty)\times\mathbb{R}^N$. We next readily deduce from (3.2) and (3.3) that, for $\tau>0$,

$$\|\nabla u_k(t)\|_\infty \le C\ \|u_0\|_\infty^{2/p}\ t^{-1/p} \le C\ \|u_0\|_\infty^{2/p}\ \tau^{-1/p} \quad\text{for}\quad t\ge\tau\,. \tag{3.9}$$

Thanks to (3.9) we may argue as in the previous step and conclude that

$$u_k \longrightarrow u \quad\text{uniformly on any compact subset of}\quad Q_\infty\,. \tag{3.10}$$

Using again the stability of continuous viscosity solutions, we deduce from the convergence (3.10) that $(t,x) \longmapsto u(t+\tau,x)$ is a viscosity solution to (1.1) with initial condition $u(\tau)$ for each $\tau>0$. In addition, denoting by $\tilde{u}_k$ the solution to the p-Laplacian equation (1.9) with initial condition $u_{0,k}$, the comparison principle entails that

$$u_k(t,x) \le \tilde{u}_k(t,x) \quad \text{for} \quad (t,x)\in Q_\infty \quad \text{and} \quad k\ge 1\,. \tag{3.11}$$

Furthermore, $(\tilde{u}_k)_{k\ge1}$ converges uniformly on any compact subset of $[0,\infty)\times\mathbb{R}^N$ towards the solution $\tilde{u}$ to the p-Laplacian equation (1.9) with initial condition u_0 [16, Ch. III]. This property and (3.11) warrant that $u(t,x)\le\tilde{u}(t,x)$ for $(t,x)\in[0,\infty)\times\mathbb{R}^N$. Recalling (3.8), we thus obtain the following inequality

$$u_k(t,x) \le u(t,x) \le \tilde{u}(t,x) \quad \text{for} \quad (t,x)\in Q_\infty \quad \text{and} \quad k\ge 1\,. \tag{3.12}$$

We then infer from (3.12) that $(u(.+1/j))_{j\ge1}$ converges towards u uniformly on any compact subset of $[0,\infty)\times\mathbb{R}^N$ as $j\to\infty$. Using once more the stability of continuous viscosity solutions, we conclude that u is a viscosity solution to (1.1), (1.2). We next argue as in the previous step to deduce from (3.3) and (3.4) that u satisfies (1.6), (1.7) and (1.8) for $t>s>0$. In addition, $u\in L^\infty(Q_\infty)$ by (1.5) and we deduce from (1.5) and (1.6) that $\|\nabla u(t)\|_\infty \le C\,\|u_0\|_\infty^{2/p}\, t^{-1/p}$ for $t>0$. Consequently, ∇u belongs to $L^{p-1}((0,T)\times B(0,R))$ for all $T>0$ and $R>0$. We then let $s\to0$ in (1.8) to conclude that $\nabla u\in L^q((0,T)\times B(0,R))$ for all $T>0$ and $R>0$ which in turn warrants that (1.8) is also valid for $s=0$.

To complete the proof of Theorem 1.1, it remains to check the uniqueness assertion for $u_0\in\mathcal{BUC}(\mathbb{R}^N)$ which actually follows at once from [18, Theorem 2.1].

4 Temporal decay estimates

This section is devoted to the proof of Proposition 1.4. Let us start with the following lemma:

Lemma 4.1 *Let u be a solution of (1.1), (1.2). If $t>s\ge0$, then*

$$\|\nabla u(t)\|_\infty \;\le\; C\,\|u(s)\|_\infty^{2/p}\,(t-s)^{-1/p}\,, \tag{4.1}$$

$$\|\nabla u(t)\|_\infty \;\le\; C\|u(s)\|_\infty^{1/q}\,(t-s)^{-1/q}\,. \tag{4.2}$$

Proof. We write

$$|\nabla u(t)| = \frac{1}{\gamma}\, u^{1-\gamma}\, |\nabla u^\gamma|$$

for $\gamma=\alpha_p$ and $\gamma=\beta_{p,q}$ and use the estimates (1.6) and (1.7). $\square$

Proof of Proposition 1.4. We first prove (1.16). Combining the Gagliardo-Nirenberg inequality, the time monotonicity of $\|u\|_1$ and the previous lemma, we obtain

$$\begin{aligned} \|u(t)\|_\infty^q &\le C\,\|\nabla u(t)\|_\infty^{qN/(N+1)}\,\|u(t)\|_1^{q/(N+1)} \\ &\le C\|\nabla u(t)\|_\infty^{qN/(N+1)}\,\|u_0\|_1^{q/(N+1)} \\ &\le C(t-s)^{-N/(N+1)}\,\|u(s)\|_\infty^{N/(N+1)}\,\|u_0\|_1^{q/(N+1)}\,. \end{aligned}$$

Integrating with respect to t over (s,∞), we obtain

$$\begin{aligned}
\tau(s) := \int_s^\infty \frac{\|u(t)\|_\infty^q}{t}\, dt &\le C\ \|u(s)\|_\infty^{N/(N+1)}\ \|u_0\|_1^{q/(N+1)} \int_s^\infty \frac{dt}{(t-s)^{N/(N+1)}t} \\
&\le C\ s^{-N/(N+1)}\ \|u_0\|_1^{q/(N+1)}\ \|u(s)\|_\infty^{N/(N+1)},
\end{aligned}$$

whence

$$\tau(s) \le C\ \|u_0\|_1^{q/(N+1)}\ \left(-\tau'(s)\right)^{N/q(N+1)}\ s^{-(N(q-1))/q(N+1)}.$$

Introducing $\tilde{\tau}(s) = \tau(s^{1/q})$ gives

$$\frac{d\tilde{\tau}}{ds}(s) + C\ \|u_0\|_1^{-q^2/N}\ \tilde{\tau}(s)^{q(N+1)/N} \le 0\,.$$

A direct computation shows that $\tilde{\tau}(s) \le C\ \|u_0\|_1^{q^2\xi}\ s^{-N\xi}$ from which we deduce that

$$\tau(s) \le C\ \|u_0\|_1^{q^2\xi}\ s^{-qN\xi}\,.$$

Now, using the time monotonicity of $\|u\|_\infty$, we obtain

$$C\ s^{-qN\xi}\ \|u_0\|_1^{q^2\xi} \ge \tau(s) \ge \int_s^{2s} \frac{\|u(t)\|_\infty^q}{t}\, dt \ge \int_s^{2s} \frac{\|u(2s)\|_\infty^q}{t}\, dt = \ln(2)\ \|u(2s)\|_\infty^q\,,$$

whence (1.16). The estimate (1.17) then readily follows from (1.16) by (4.2). A similar proof relying on (4.1) gives the estimates (1.18) and (1.19). □

5 Limit values of $\|u(t)\|_1$

In this section we investigate the possible values of the limit as $t \to \infty$ of the L^1-norm of non-negative solutions to (1.1), (1.2) and prove Proposition 1.5. We first show that, if q is small enough, the dissipation mechanism induced by the nonlinear absorption term is sufficiently strong to drive the L^1-norm of u to zero in infinite time.

Proposition 5.1 *If $q \in (1, q_*]$ then*

$$\lim_{t\to\infty} \|u(t)\|_1 = 0\,.$$

Proof. It first follows from the integration of (1.1) over $(0,t) \times \mathbb{R}^N$ that

$$\|u(t)\|_1 + \int_0^t \|\nabla u(s)\|_q^q\, ds = \|u_0\|_1\,, \tag{5.1}$$

which readily implies that $t \longmapsto \|\nabla u(t)\|_q^q$ belongs to $L^1(0,\infty)$. Consequently,

$$\omega(t) := \int_t^\infty \|\nabla u(s)\|_q^q\, ds \underset{t\to\infty}{\longrightarrow} 0\,. \tag{5.2}$$

We next consider a $\mathcal{C}^\infty$-smooth function ϑ in $\mathbb{R}^N$ such that $0 \le \vartheta \le 1$ and

$$\vartheta(x) = 0 \quad \text{if} \quad |x| \le 1/2 \quad \text{and} \quad \vartheta(x) = 1 \quad \text{if} \quad |x| \ge 1\,.$$

For $R > 0$ and $x \in \mathbb{R}^N$ we put $\vartheta_R(x) = \vartheta(x/R)$. We multiply (1.1) by $\vartheta_R(x)$ and integrate over $(t_1, t_2) \times \mathbb{R}^N$ to obtain

$$\int_{\mathbb{R}^N} u(t_2,x)\,\vartheta_R(x)\,dx \le \int_{\mathbb{R}^N} u(t_1,x)\,\vartheta_R(x)\,dx + \frac{1}{R}\int_{t_1}^{t_2} |\nabla u(s,x)|^{p-2}\,\nabla\vartheta\left(\frac{x}{R}\right)\,\nabla u(s,x)\,dxds\,,$$

which, together with the properties of ϑ, gives

$$\int_{\{|x|\ge 2R\}} u(t_2,x)\,dx \le \int_{\{|x|\ge R\}} u(t_1,x)\,dx + \frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \left|\nabla\vartheta\left(\frac{x}{R}\right)\right|\,|\nabla u(s,x)|^{p-1}\,dxds\,. \quad (5.3)$$

Case 1: $q \in [p-1, q_*]$. By the Hölder inequality we have

$$\begin{aligned}
& \frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \left|\nabla\vartheta\left(\frac{x}{R}\right)\right|\,|\nabla u(s,x)|^{p-1}\,dxds \\
\le\ & R^{(N(q-p+1)-q)/q}\,(t_2-t_1)^{(q-p+1)/q}\,\|\nabla\vartheta\|_{(q-p+1)/q}\,\left(\int_{t_1}^{t_2}\|\nabla u(s)\|_q^q\,dxds\right)^{(p-1)/q} \\
\le\ & C\,R^{(N(q-p+1)-q)/q}\,\omega(t_1)^{(p-1)/q}\,(t_2-t_1)^{(q-p+1)/q}\,.
\end{aligned}$$

Combining the above inequality with (1.16), (5.3) and the time monotonicity of $\|u\|_1$ we obtain

$$\begin{aligned}
\|u(t_2)\|_1 &= \int_{\{|x|\le 2R\}} u(t_2,x)\,dx + \int_{\{|x|\ge 2R\}} u(t_2,x)\,dx \\
&\le C\,R^N\,\|u(t_2)\|_\infty + \int_{\{|x|\ge R\}} u(t_1,x)\,dx \\
&+ C\,R^{(N(q-p+1)-q)/q}\,\omega(t_1)^{(p-1)/q}\,(t_2-t_1)^{(q-p+1)/q} \\
&\le \int_{\{|x|\ge R\}} u(t_1,x)\,dx + C\,R^N\,(t_2-t_1)^{-N\xi} \\
&+ C\,R^{(N(q-p+1)-q)/q}\,\omega(t_1)^{(p-1)/q}\,(t_2-t_1)^{(q-p+1)/q}\,.
\end{aligned}$$

Choosing

$$R = R(t_1,t_2) := \omega(t_1)^{(p-1)/(q+N(p-1))}\,(t_2-t_1)^{(qN\xi+q-p+1)/(q+N(p-1))}$$

we are led to

$$\begin{aligned}
\|u(t_2)\|_1 &\le \int_{\{|x|\ge R(t_1,t_2)\}} u(t_1,x)\,dx \\
&+ C\,\omega(t_1)^{(N(p-1))/(q+N(p-1))}\,(t_2-t_1)^{-qN\xi(N+1)(q_*-q)/(q+N(p-1))}\,.
\end{aligned}$$

Since $\xi > 0$ and $q_* - q > 0$ we may let $t_2 \to \infty$ in the previous inequality to conclude that

$$\begin{aligned} I_1(\infty) &\le 0 \quad \text{if} \quad q \in [p-1, q_*)\,, \\ I_1(\infty) &\le C\ \omega(t_1)^{(N(p-1))/(q_*+N(p-1))} \quad \text{if} \quad q = q_*\,. \end{aligned}$$

We have used here that $R(t_1,t_2) \to \infty$ as $t_2 \to \infty$ and that $u(t_1) \in L^1(\mathbb{R}^N)$. Owing to the non-negativity of $I_1(\infty)$, we readily obtain that $I_1(\infty) = 0$ if $q \in [p-1, q_*)$. When $q = q_*$, we let $t_1 \to \infty$ and use (5.2) to conclude that $I_1(\infty) = 0$ also in that case.
Case 2: $q \in (1, p-1)$. By (1.17) and (5.3) we have

$$\begin{aligned} \int_{\{|x|\ge 2R\}} u(t_2,x)\ dx &\le \int_{\{|x|\ge R\}} u(t_1,x)\ dx + \frac{1}{R}\ \|\nabla\vartheta\|_\infty \int_{t_1}^{t_2} \|\nabla u(s)\|_\infty^{p-1-q}\ \|\nabla u(s)\|_q^q\ ds \\ &\le \int_{\{|x|\ge R\}} u(t_1,x)\ dx + \frac{C}{R} \int_{t_1}^{t_2} s^{-(p-1-q)(N+1)\xi}\ \|\nabla u(s)\|_q^q\ ds \\ &\le \int_{\{|x|\ge R\}} u(t_1,x)\ dx + \frac{C}{R}\ t_1^{-(p-1-q)(N+1)\xi}\ \omega(t_1)\,. \end{aligned}$$

Taking $t_1 = 1$ and noting that $\omega(t_1) \le \omega(0) \le \|u_0\|_1$, we end up with

$$\int_{\{|x|\ge 2R\}} u(t_2,x)\ dx \le \int_{\{|x|\ge R\}} u(1,x)\ dx + \frac{C}{R}\,, \quad t_2 \ge 1\,.$$

We then infer from (1.16) and the above inequality that, if $t_2 \ge 1$,

$$\|u(t_2)\|_1 \le C\ R^N\ t^{-N\xi} + \int_{\{|x|\ge R\}} u(1,x)\ dx + \frac{C}{R}$$

and the choice $R = R(t_2) = t_2^{(N\xi)/(N+1)}$ gives

$$\|u(t_2)\|_1 \le \int_{\{|x|\ge R(t_2)\}} u(1,x)\ dx + C\ t_2^{-(N\xi)/(N+1)}\,.$$

Since $R(t_2) \to \infty$ as $t_2 \to \infty$ and $u(1) \in L^1(\mathbb{R}^N)$ we may let $t_2 \to \infty$ in the above inequality to establish that $I_1(\infty) = 0$, which completes the proof of Proposition 5.1. $\square$

We next turn to higher values of q and adapt an argument of [5, Theorem 6] to show the positivity of $I_1(\infty)$.

Proposition 5.2 *Assume that* $\|u_0\|_1 > 0$ *and* $q > q_*$. *Then* $I_1(\infty) > 0$.

Proof. Since $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ is not identically equal to zero there are $x_0 \in \mathbb{R}^N$ and a radially symmetric and non-increasing continuous function $U_0 \not\equiv 0$ such that $u_0(x) \ge U_0(x - x_0)$. Denoting by U the solution to (1.1) with initial condition U_0 it follows from the invariance of (1.1) by translation and the comparison principle that

$$u(t,x) \ge U(t, x - x_0)\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N\,. \tag{5.4}$$

Let $\tau > 0$ and $x \in \mathbb{R}^N$. Since

$$\nabla U(\tau,x) = \frac{p-1}{p-2}\ U(\tau,x)^{1/(p-1)}\ \nabla\left(U^{(p-2)/(p-1)}\right)(\tau,x)$$

and $q > q_* > p-1$, we infer from (1.11) and the time monotonicity of $\|u\|_\infty$ that

$$\begin{aligned}
|\nabla U(\tau,x)|^q &\leq \left(\frac{p-1}{p-2}\right)^q\ U(\tau,x)^{q/(p-1)}\ \left|\nabla\left(U^{(p-2)/(p-1)}\right)(\tau,x)\right|^q \\
&\leq C\ U(\tau,x)\ \|U(\tau)\|_\infty^{(q-p+1)/(p-1)}\ \left\|U\left(\frac{\tau}{2}\right)\right\|_\infty^{q(p-2)/p(p-1)}\ \tau^{-q/p} \\
&\leq C\ U(\tau,x)\ \left\|U\left(\frac{\tau}{2}\right)\right\|_\infty^{(2q-p)/p}\ \tau^{-q/p}\,,
\end{aligned}$$

whence, by (1.18),

$$|\nabla U(\tau,x)|^q \leq C\ U(\tau,x)\ \tau^{-\eta/\xi}\,. \tag{5.5}$$

Consider now $s \in (0,\infty)$ and $t \in (s,\infty)$. It follows from (1.1) and (5.5) that

$$\begin{aligned}
\|U(t)\|_1 &= \|U(s)\|_1 - \int_s^t \int_{\mathbb{R}^N} |\nabla U(\tau,x)|^q\ dxd\tau \\
&\geq \|U(s)\|_1 - C\ \int_s^t \tau^{-\eta/\xi}\ \|U(\tau)\|_1\ d\tau\,.
\end{aligned}$$

Owing to the monotonicity of $\tau \longmapsto \|U(\tau)\|_1$, we further obtain

$$\|U(t)\|_1 \geq \|U(s)\|_1\ \left(1 - C\ \int_s^t \tau^{-\eta/\xi}\ d\tau\right)\,.$$

Since $q > q_*$ we have $\eta > \xi$ and the right-hand side of the above inequality has a finite limit as $t \to \infty$. We may then let $t \to \infty$ to obtain

$$\mathcal{I}_1(\infty) := \lim_{t\to\infty} \|U(t)\|_1 \geq \|U(s)\|_1\ \left(1 - C\ s^{-(\eta-\xi)/\xi}\right)\,,\quad s > 0\,.$$

Consequently, for s large enough, we have $\mathcal{I}_1(\infty) \geq \|U(s)\|_1/2$, while [1, Lemma 4.1] warrants that $\|U(s)\|_1 > 0$ for each $s \geq 0$ since $U_0 \not\equiv 0$. Therefore, $\mathcal{I}_1(\infty) > 0$. Recalling (5.4) we realize that $\|u(t)\|_1 \geq \|U(t)\|_1$ for each $t \geq 0$ so that $I_1(\infty) \geq \mathcal{I}_1(\infty) > 0$. □

6 Compactly supported initial data

This section is devoted to the proofs of Theorem 1.6 and Corollary 1.7. Let $u_0 \in L^1(\mathbb{R}^N) \cap \mathcal{BC}(\mathbb{R}^N)$ be a non-negative initial condition with compact support in the ball $B(0,R_0)$ for some $R_0 > 0$. Denoting by u the corresponding solution to (1.1), (1.2) and by v the corresponding solution to the p-Laplacian equation

$$\partial_t v - \Delta_p v = 0\ ,\quad (t,x) \in Q_\infty\,, \tag{6.1}$$

with initial condition $v(0)=u_0$, the comparison principle ensures that

$$0 \le u(t,x) \le v(t,x)\,, \quad (t,x)\in Q_\infty\,. \tag{6.2}$$

Since u_0 is compactly supported, so is $v(t)$ for each $t\ge 0$ by [16, Lemma 8.1] and $\,\mathrm{Supp}\, v(t)\subset B(0,C_1 t^\eta)$. Consequently, $u(t)$ is compactly supported for each $t\ge 0$ with $\,\mathrm{Supp}\, u(t)\subset B(0,C_1 t^\eta)$. In particular, the support of u does not expand faster than that of v with time. A natural question is then whether the damping term slows down this expansion and the answer depends heavily on the value of q. We shall thus distinguish between three cases in the proof of Theorem 1.6.

We first note that, since u_0 is non-negative continuous and compactly supported, there exists a non-negative continuous radially symmetric and non-increasing function U_0 with compact support such that $0\le u_0\le U_0$. Denoting by U the corresponding solution to (1.1) with initial condition $U(0)=U_0$, the function $x\longmapsto U(t,x)$ is also radially symmetric and non-increasing for each $t\ge 0$ and we deduce from the comparison principle that

$$0 \le u(t,x) \le U(t,x)\,, \quad (t,x)\in [0,\infty)\times\mathbb{R}^N\,. \tag{6.3}$$

Moreover, by comparison with the p-Laplacian equation, $U(t)$ is also compactly supported for each $t\ge 0$ with $\,\mathrm{Supp}\, U(t)\subset B(0,\sigma(t))$ for some $\sigma(t)>0$. Clearly,

$$\varrho(t)\le \sigma(t)\,, \quad t\ge 0\,, \tag{6.4}$$

by (6.3).

It next follows from (1.1) that, if y is a non-negative function in $\mathcal{C}^1([0,\infty))$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\{|x|\ge y(t)\}} U(t,x)\ dx &= \int_{\{|x|\ge y(t)\}} \partial_t U(t,x)\ dx - y'(t)\int_{\{|x|=y(t)\}} U(t,x)\ dx\\
&\le \int_{\{|x|\ge y(t)\}} \mathrm{div}\ \left(|\nabla U|^{p-2}\ \nabla U\right)(t,x)\ dx\\
&- y'(t)\int_{\{|x|=y(t)\}} U(t,x)\ dx\\
&\le -\int_{\{|x|=y(t)\}} |\nabla U(t,x)|^{p-2}\ \nabla U(t,x)\cdot \frac{x}{|x|}\ dx\\
&- y'(t)\int_{\{|x|=y(t)\}} U(t,x)\ dx\\
&\le \int_{\{|x|=y(t)\}} \left\{|\nabla U(t,x)|^{p-1} - y'(t)\ U(t,x)\right\}\ dx\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\int_{\{|x|\ge y(t)\}} U(t,x)\ dx\\
\le\ &\int_{\{|x|=y(t)\}} \left\{\frac{p-1}{p-2}\ \left|\nabla\left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} - y'(t)\right\}\ U(t,x)\ dx\,.
\end{aligned} \tag{6.5}$$

The next step is to use the gradient estimates established in Theorem 1.2 to find a suitable function y for which the right-hand side of (6.5) is non-positive. The gradient estimates depending on the value of q, we handle separately the cases $q \in (1, p-1]$ and $q \in (p-1, q_*)$.

Proof of Theorem 1.6: $q \in (1, p-1]$**.** In that case we infer from (1.13) and (1.16) that

$$\begin{aligned}\left|\nabla \left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\leq C \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(p-1-q)/q} t^{-(p-1)/q} \\ &\leq C\, t^{-\xi((p-1)(N+1)-N)},\end{aligned}$$

so that (6.5) becomes

$$\frac{d}{dt}\int_{\{|x|\geq y(t)\}} U(t,x)\ dx \leq \int_{\{|x|=y(t)\}} \left\{C\ t^{-\xi((p-1)(N+1)-N)} - y'(t)\right\}\ U(t,x)\ dx$$

Choosing $y'(t) := C\ t^{-\xi((p-1)(N+1)-N)}$ for $t \geq 1$ and $y(1) = \sigma(1)$, we conclude that

$$\int_{\{|x|\geq y(t)\}} U(t,x)\ dx \leq \int_{\{|x|\geq \sigma(1)\}} U(1,x)\ dx = 0$$

for $t \geq 1$. Consequently, $\sigma(t) \leq y(t)$ for $t \geq 1$ from which we deduce that $\varrho(t) \leq y(t)$ for $t \geq 1$ by (6.3). Now, either $q \in (1, p-1)$ and $\xi((p-1)(N+1)-N) > 1$. Therefore $y(t)$ has a finite limit as $t \to \infty$ from which (1.22) readily follows. Or $q = p-1$ and $y(t) = \sigma(1) + C\ \ln t$ which gives (1.23). □

We next consider the case $q \in (p-1, q_*)$ which turns out to be more complicated as (1.13) is no longer available. We instead use (1.11) which somehow provides less information and thus complicates the proof. We shall also need the following lemma which is an easy consequence of the Poincaré and Hölder inequalities.

Lemma 6.1 *There is a positive constant κ depending only on N and q such that, if $R > 0$ and w is a function in $W_0^{1,q}(B(0,R))$ then*

$$R^{-1/\xi}\ \|w\|_{L^1(B(0,R))}^q \leq \kappa\ \|\nabla w\|_{L^q(B(0,R))}^q\,. \tag{6.6}$$

Proof of Theorem 1.6: $q \in (p-1, q_*)$**.** We fix $t_0 \geq 0$. It follows from (1.11) and (1.16) that

$$\begin{aligned}\frac{p-1}{p-2}\ \left|\nabla \left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\leq C \left\| u\left(\frac{t+t_0}{2}\right)\right\|_\infty^{(p-2)/p} (t-t_0)^{-(p-1)/p} \\ &\leq C\ \|u(t_0)\|_1^{q\xi(p-2)/p}\ (t-t_0)^{-(p-1+N\xi(p-2))/p}\end{aligned}$$

for $t \geq t_0$. Since $q > p-1 > N(p-1)/(N+1)$, we have $1 - N\xi(p-2) > 0$ and we choose $y(t) = \sigma(t_0) + pC\ \|u(t_0)\|_1^{q\xi(p-2)/p}\ (t-t_0)^{(1-N\xi(p-2))/p}/(1-N\xi(p-2))$ for $t \geq t_0$. The previous inequality then reads

$$\frac{p-1}{p-2}\ \left|\nabla \left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} \leq y'(t)\,, \quad t \geq t_0\,.$$

Combining the latter estimate with (6.5) we realize that

$$\frac{d}{dt}\int_{\{|x|\ge y(t)\}} U(t,x)\ dx \le 0 \quad \text{for} \quad t\ge t_0\,,$$

whence

$$\int_{\{|x|\ge y(t)\}} U(t,x)\ dx \le \int_{\{|x|\ge \sigma(t_0)\}} U(t_0,x)\ dx = 0\,, \quad t\ge t_0\,.$$

We have thus established that $\sigma(t)\le y(t)$ for $t\ge t_0$ from which we readily conclude that

$$\sigma(t)\le \sigma(t_0) + C\ \|U(t_0)\|_1^{q\xi(p-2)/p}\ (t-t_0)^{(1-N\xi(p-2))/p}\,, \quad t\ge t_0\,. \tag{6.7}$$

We next integrate (1.1) over $\mathbb{R}^N$ and obtain

$$\frac{d}{dt}\|U(t)\|_1 + \|\nabla U(t)\|_q^q = 0\,.$$

Since the support of $U(t)$ is included in $B(0,\sigma(t))$, we infer from Lemma 6.1 that

$$\|\nabla U(t)\|_q^q = \int_{\{|x|<\sigma(t)\}} |\nabla U(t,x)|^q\ dx \ge \frac{1}{\kappa\ \sigma(t)^{1/\xi}}\ \left(\int_{\{|x|<\sigma(t)\}} U(t,x)\ dx\right)^q = \frac{\|U(t)\|_1^q}{\kappa\ \sigma(t)^{1/\xi}}\,.$$

Inserting this lower bound in the previous diferential equality gives

$$\frac{d}{dt}\|U(t)\|_1 + \frac{1}{\kappa}\ \frac{\|U(t)\|_1^q}{\sigma(t)^{1/\xi}} \le 0\,. \tag{6.8}$$

Before going on we introduce the following notations:

$$\Sigma(T) := \sup_{t\in[1,T]}\left\{t^{-A}\ \sigma(t)\right\}, \qquad A := \frac{q-p+1}{2q-p}\,,$$

$$L(T) := \sup_{t\in[1,T]}\left\{t^{B}\ \|U(t)\|_1\right\}, \qquad B := \frac{(N+1)(q_*-q)}{2q-p}\,,$$

for $T\ge 1$ and notice that $\Sigma(T)$ and $L(T)$ are well-defined for each $T\ge 1$ while A and B satisfy

$$A + \frac{q\xi(p-2)}{p}\ B = \frac{1-N\xi(p-2)}{p} \quad \text{and} \quad 1-\frac{A}{\xi} = (q-1)\ B\,. \tag{6.9}$$

Fix $T\ge 1$. We infer from (6.8) that, if $t\in[1,T]$,

$$\frac{d}{dt}\|U(t)\|_1 + \frac{t^{-A/\xi}}{\kappa}\ \frac{\|U(t)\|_1^q}{t^{-A/\xi}\ \sigma(t)^{1/\xi}} \le 0$$

$$\frac{d}{dt}\|U(t)\|_1 + \frac{1}{\kappa\ \Sigma(T)^{1/\xi}}\ \frac{\|U(t)\|_1^q}{t^{A/\xi}} \le 0\,,$$

which gives

$$\|U(t)\|_1 \le C\ \Sigma(T)^{1/((q-1)\xi)}\ \left(t^{(q-1)B}-1\right)^{-1/(q-1)}\ , \quad t\in[1,T]\,, \tag{6.10}$$

after integration. Consider next $t\in[1,T]$. Either $t\le 4$ and it follows from (6.7) with $t_0=1$ that

$$t^{-A}\ \sigma(t) \le t^{-A}\ \sigma(1) + C\ \|U(1)\|_1^{q\xi(p-2)/p}\ (t-1)^{(1-N\xi(p-2))/p}\ t^{-A} \le C\,.$$

Or $t\ge 4$ and we infer from (6.7) with $t_0=t/2\ge 2$, (6.9) and (6.10) that

$$\begin{aligned} t^{-A}\ \sigma(t) &\le t^{-A}\ \sigma\left(\frac{t}{2}\right) + C\ \left\|U\left(\frac{t}{2}\right)\right\|_1^{q\xi(p-2)/p}\ t^{q\xi(p-2)B/p} \\ &\le 2^{-A}\ \Sigma(T) + C\ \Sigma(T)^{(q(p-2))/(p(q-1))}\,. \end{aligned}$$

Consequently,

$$t^{-A}\ \sigma(t) \le 2^{-A}\ \Sigma(T) + C\ \left(1+\Sigma(T)^{(q(p-2))/(p(q-1))}\right)\ , \quad t\in[1,T]\,,$$

from which we conclude that

$$\Sigma(T) \le 2^{-A}\ \Sigma(T) + C\ \left(1+\Sigma(T)^{(q(p-2))/(p(q-1))}\right)\ .$$

Since $A>0$ and $q(p-2)<p(q-1)$ the above inequality entails that $\Sigma(T)\le C$ for each $T\ge 1$, the constant C being independent of T. Recalling (6.4) we have thus proved that $\varrho(t)\le\sigma(t)\le C\ t^A$ for $t\ge 1$, hence (1.24).

Furthermore the boundedness of $\Sigma(T)$ and (6.10) ensure that $\|U(t)\|_1\le C\ (t-1)^{-B}$ for $t\ge 1$ which, together with (6.3), implies that

$$\|u(t)\|_1 \le C\ t^{-B}\ , \quad t\ge 2\,. \tag{6.11}$$

We have thus also established the assertion (iii) of Corollary 1.7. □

Proof of Corollary 1.7. Assume first that $q\in(1,p-1)$. Then, on the one hand, it follows from (1.22) that there is $\varrho_\infty>0$ such that $\varrho(t)\le\varrho_\infty$ for $t\ge 1$. On the other hand, we may proceed as in the proof of (6.8) to establish that

$$\frac{d}{dt}\|u(t)\|_1 + \frac{1}{\kappa}\ \frac{\|u(t)\|_1^q}{\varrho(t)^{1/\xi}} \le 0\,. \tag{6.12}$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_1 + \frac{1}{\kappa}\ \frac{\|u(t)\|_1^q}{\varrho_\infty^{1/\xi}} \le 0\,, \quad t\ge 1\,,$$

from which (1.26) readily follows.

Similarly, if $q=p-1$, we infer from (1.23) and (6.12) that, for $t\ge 2$,

$$
\begin{aligned}
\|u(t)\|_1 &\le C\ \left(\int_1^t (1+\ln s)^{-1/\xi}\ ds\right)^{-1/(q-1)} \\
&\le C\ \left(\int_0^{\ln t} (1+s)^{-1/\xi}\ e^s\ ds\right)^{-1/(q-1)} \\
&\le C\ \left((1+\ln t)^{-1/\xi}\ (t-1)\right)^{-1/(q-1)}\ ,
\end{aligned}
$$

which gives (1.27).

Since the case $q\in(p-1,q_*)$ has already been handled in the proof of Theorem 1.6 (recall (6.11)) we are left with the case $q=q_*$. In that particular case, $\xi=\eta$ and we infer from (1.25) and (6.12) that

$$
\frac{d}{dt}\|u(t)\|_1+\frac{C}{t}\ \|u(t)\|_1^q\le 0\,,\quad t\ge 1\,,
$$

which gives (1.29) by integration. ∎

7 Persistence of dead cores

Proof of Proposition 1.8. We first study the one-dimensional case $N=1$. We consider a non-negative function $y\in\mathcal{C}^1([0,\infty))$ to be specified later and proceed as in the proof of Theorem 1.6 to deduce from (1.1) that

$$
\frac{d}{dt}\int_{-y(t)}^{y(t)} u(t,x)\ dx=\left[\left(\frac{p-1}{p-2}\ \left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1}+y'(t)\right)\ u(t,x)\right]_{x=-y(t)}^{x=y(t)} \tag{7.13}
$$

On the one hand we infer from (1.6) that

$$
\begin{aligned}
\frac{p-1}{p-2}\ \left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\le \frac{p-1}{p-2}\,C(p,1)^{p-1}\ \|u_0\|_\infty^{(p-2)/p}\ t^{-(p-1)/p} \\
&\le c_1\ \|u_0\|_\infty^{(p-2)/p}\ t^{-(p-1)/p}\,.
\end{aligned}
$$

On the other hand, since $p-1>q$, we have $\beta_{p,q}=\alpha_p=(p-2)/(p-1)$ and it follows from (1.7) that

$$
\begin{aligned}
\frac{p-1}{p-2}\ \left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\le \frac{p-1}{p-2}\,C(p,q,1)^{p-1}\ \|u_0\|_\infty^{(p-1-q)/q}\ t^{-(p-1)/q} \\
&\le c_2\ \|u_0\|_\infty^{(p-1-q)/q}\ t^{-(p-1)/q}\,.
\end{aligned}
$$

Consequently, choosing

$$
\left\{
\begin{aligned}
& y'(t)=-\min\left\{c_1\ \|u_0\|_\infty^{p-2}\ t^{-(p-1)/p}, c_2\ \|u_0\|_\infty^{(p-1-q)/q}\ t^{-(p-1)/q}\right\}, \\
& y(0)=R_0\,,
\end{aligned}
\right. \tag{7.14}
$$

we have

$$\frac{p-1}{p-2}\ \left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} \le -y'(t)\,. \tag{7.15}$$

We then deduce from (7.13) and (7.15) that

$$\frac{d}{dt}\int_{-y(t)}^{y(t)} u(t,x)\ dx \le 0\,,$$

whence

$$\int_{-y(t)}^{y(t)} u(t,x)\ dx \le \int_{-R_0}^{R_0} u_0(x)\ dx = 0 \quad \text{for} \quad t\ge 0\,.$$

Now it is actually possible to compute the function y defined by (7.14) and to see that

$$y(t) \ge y_\infty := \lim_{s\to\infty} y(s) = R_0 - \delta_0\ \|u_0\|_\infty^{(p-1-q)/(p-q)}$$

for some δ_0 depending only on c_1, c_2, p, and q. Then $u(t,x)=0$ for $x\in[-y_\infty,y_\infty]$ and $t\ge 0$, and $y_\infty>0$ under the assumptions of Proposition 1.8.

In several space dimensions $N\ge 2$, consider $\varepsilon\in(0,R_0/2)$ and put

$$u_0^\varepsilon(x_1) := \begin{cases} \|u_0\|_\infty & \text{if} \quad |x_1|\ge R_0\,, \\ \dfrac{\|u_0\|_\infty}{\varepsilon}\ (|x_1|-R_0+\varepsilon) & \text{if} \quad R_0-\varepsilon\le|x_1|\le R_0\,, \\ 0 & \text{if} \quad |x_1|\le R_0-\varepsilon\,, \end{cases}$$

Clearly, $u_0 \le u_0^\varepsilon$ in $\mathbb{R}^N$ and the comparison principle entails that $u(t,x_1,x_2,\ldots,x_N) \le u^\varepsilon(t,x_1)$ for $(t,x)\in[0,\infty)\times\mathbb{R}^N$, where u^ε denotes the solution to (1.1) with initial condition u_0^ε and $N=1$. Choosing ε appropriately small provides the expected result in the x_1-direction. We proceed analogously in every direction to complete the proof of Proposition 1.8. $\square$

A Proof of Lemma 2.1

Since $\partial_t u = \varphi'(v)\ \partial_t v$ and $\nabla u = \varphi'(v)\ \nabla v$ we deduce from (2.1) that

$$\partial_t v - a\ \Delta v - a\ \frac{\varphi''}{\varphi'}\ w - 2\ a'\ \varphi'\ \varphi'' w^2 - 2\ a'\ \varphi'^2\ (\nabla v)^t D^2 v \nabla v + \frac{b'}{\varphi'} = 0.$$

Observing that

$$(\nabla v)^t D^2 v \nabla v = \frac{1}{2}\ \nabla v\cdot\nabla w \quad \text{and} \quad \Delta w = 2\ \nabla v\cdot\nabla\Delta v + 2\ \sum_{i,j} |\partial_i\partial_j v|^2\ ,$$

elementary, but laborious calculation shows that

$$\partial_t w - \mathcal{A} w + 2\ a\ \sum_{i,j} |\partial_i \partial_j v|^2 + 2\ a'\ \varphi'\ \varphi''\ w\ \nabla v \cdot \nabla w - \mathcal{V} \cdot \nabla w + 2\ \mathcal{S}_1\ w^2 + 2\ \mathcal{R}_2\ w = 0$$

with

$$\mathcal{S}_1 := -a\ \left(\frac{\varphi''}{\varphi'}\right)' - 2\ a'\ \varphi'\ \varphi''\ \Delta v - 4\ a''\,(\varphi'\varphi'')^2\ w^2 - 2\ a'\ w\left(2\varphi''^2 + \varphi'\varphi'''\right)\,, \tag{A.1}$$

and

$$\begin{aligned} \mathcal{V} \ &:= \ 2\ \left[a\ \frac{\varphi''}{\varphi'} + a'\ \varphi'^2\ \left(\Delta v + \frac{2\varphi''}{\varphi'}\ w\right)\right]\ \nabla v \\ &+ \ 4\ \varphi'\ \varphi''\left[\left(a''\ \varphi'^2\ w + 3\ a'\right) + a''\ \varphi'^2\ w\right]\ w\ \nabla v \\ &+ \ 2\ \left[a''\ \varphi'^4\ \nabla v \cdot \nabla w - b'\ \varphi'\right]\ \nabla v + a'\ \varphi'^2\ \nabla w\,. \end{aligned} \tag{A.2}$$

In order to handle the term involving Δv in $\mathcal{S}_1$ we proceed as in [10]: more precisely we have

$$\begin{aligned} & 2\ a\ \sum_{i,j} |\partial_i\partial_j v|^2 + 2\ a'\ \varphi'\ \varphi''\ w\ \nabla v \cdot \nabla w - 4\ a'\ \varphi'\ \varphi''\ \Delta v\ w^2 \\ = \ & 4\ a'\ \varphi'\ \varphi''\ w\left(\frac{1}{2}\ \nabla v \cdot \nabla w - w\ \Delta v\right) + 2\ a\ \sum_{i,j} |\partial_i\partial_j v|^2 \\ = \ & 4\ a'\ \varphi'\ \varphi''\ w\left(\sum_{i,j} \partial_i\partial_j v\ \partial_i v\ \partial_j v - w\ \sum_i \partial_i^2 v\right) + 2\ a\ \sum_{i,j} |\partial_i\partial_j v|^2 \\ = \ & \sum_i \left\{2\ a\ \left|\partial_i^2 v\right|^2 + 4\ a'\ \varphi'\ \varphi''\ w\ \left(|\partial_i v|^2 - w\right)\ \partial_i^2 v\right\} \\ + \ & \sum_{i \neq j} \left\{2\ a\ |\partial_i\partial_j v|^2 + 4\ a'\ \varphi'\ \varphi''\ w \partial_i\partial_j v\ \partial_i v\ \partial_j v\right\} \\ = \ & 2\ a\ \sum_i \left\{\partial_i^2 v + \frac{a'}{a}\ \varphi'\ \varphi''\ w\left(|\partial_i v|^2 - w\right)\right\}^2 \\ - \ & 2\ \sum_i \frac{a'^2}{a}\ (\varphi'\varphi'')^2\ w^2\ \left(|\partial_i v|^2 - w\right)^2 \\ + \ & 2\ a\ \sum_{i \neq j} \left\{\partial_i\partial_j v + \frac{a'}{a}\ \varphi'\ \varphi''\ w\ \partial_i v\ \partial_j v\right\}^2 \\ - \ & 2\ \sum_{i \neq j} \frac{a'^2}{a}\ (\varphi'\varphi'')^2\ w^2\ |\partial_i v|^2\ |\partial_j v|^2 \\ \geq \ & -2\ (N-1)\ \frac{a'^2}{a}\ (\varphi'\varphi'')^2\ w^2\,. \end{aligned}$$

Consequently,

$$2\ a\ \sum_{i,j} |\partial_i \partial_j v|^2 + 2\ a'\ \varphi'\ \varphi''\ w\ \nabla v \cdot \nabla w + 2\ \mathcal{S}_1\ w^2 \ge 2\ \mathcal{R}_1\ w^2\,,$$

which completes the proof of the first assertion of Lemma 2.1.

In the case where $x \longmapsto u(t,x)$ is radially symmetric and non-increasing for each $t \ge 0$, we have $u(t,x) = U(t,|x|)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and $\partial_r U(t,r) \le 0$ for $(t,r) \in [0,\infty) \times [0,\infty)$. Introducing $V = \varphi^{-1}(U)$ we have $v(t,x) = V(t,|x|)$ and the monotonicity of φ warrants that $\partial_r V(t,r) \le 0$. In addition, owing to the non-negativity of a', φ' and φ'', we have

$$\begin{aligned} & 2\ a'\ \varphi'\ \varphi''\ w\ \nabla v \cdot \nabla w - 4\ a'\ \varphi'\ \varphi''\ w^2\ \Delta v \\ = \ & 2\ a'\ \varphi'\ \varphi''\ w\ \left[2\ |\partial_r V|^2\ \partial_r^2 V - 2\ |\partial_r V|^2\ \left(\partial_r^2 V + \frac{N-1}{r}\ \partial_r V \right) \right] \\ \ge \ & 0\,, \end{aligned}$$

from which we deduce that

$$2\ a'\ \varphi'\ \varphi''\ w\ \nabla v \cdot \nabla w + 2\ \mathcal{S}_1\ w^2 \ge 2\ \mathcal{R}_1^r\ w^2\,,$$

and end the proof of Lemma 2.1. □

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